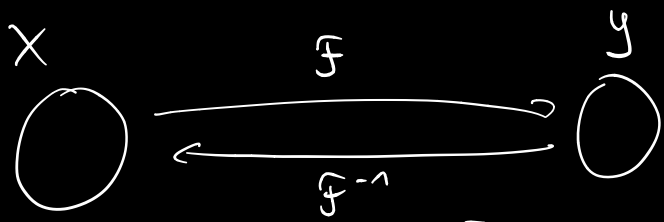
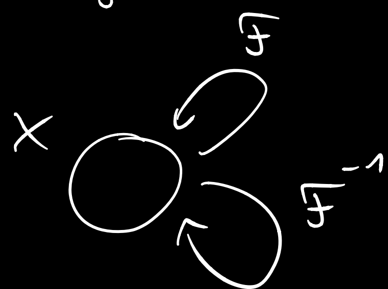
Übungsstunde 7:

Themen heute:

- Iso- & Automorphismus
- Darstellungsmatrizen zu linearen Abbildungen
- Kommutative Diagramme
- Koordinatenabbildung
- Abbildungsmatrizen unter einem Basiswechsel
- Verkettung linearer Operationen

Isomorphismus/Automorphismus:

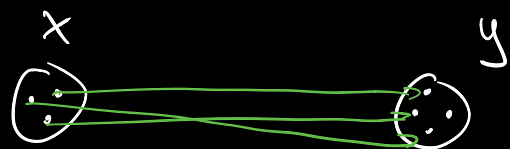




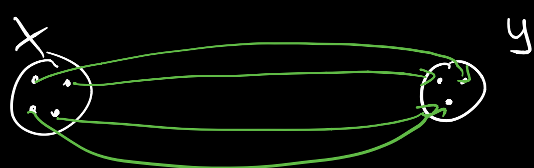
Isomorphismus: Bijektive Abb. von X nach Y
Automorphismus: " " " X nach X

Rep. Bijektiv:

Injektivität:



Surjektivität:



$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$ ⇝ keine Isomorphe Abb.

Abbildungsmatrizen / Darstellungsmatrizen:

Beispiel: $V = \mathcal{P}_2$, $W = \mathcal{P}_1$ $\qquad F: V \to W$, $\boxed{p(x)} \mapsto \boxed{p'(x) + p''(x)}$

$V: \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$, $W: \begin{bmatrix} 1 \\ x \end{bmatrix}$ $\qquad \underbrace{B = \{ b^{(1)} = 1, b^{(2)} = x, b^{(3)} = x^2 \}}$ $\begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix}$

$4x^2 + 3x + 1 \longleftrightarrow \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$$\begin{aligned} F(1) &= 0 &&= \boxed{0 \cdot 1 + 0 \cdot x} \\ F(x) &= 1 &&= \boxed{1 \cdot 1 + 0 \cdot x} \\ F(x^2) &= \underbrace{2x + 2}_{\text{Das ist neu!}} &&= \boxed{2 \cdot 1 + 2 \cdot x} \end{aligned} \qquad \Rightarrow \underline{\underline{A}} = \underline{\underline{\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}}}$$

Bsp:

$$\begin{tikzcd} x^2 + x + 1 \arrow[r, "F"] \arrow[d, "k_x"'] & 2x + 1 + 2 = \underline{2x + 3} \\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \arrow[r, "\underline{\underline{A}}"] & \begin{bmatrix} 3 \\ 2 \end{bmatrix} \arrow[u, "k_x^{-1}"'] \end{tikzcd}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Kommutative Diagramme:

<u>Def:</u> Ein kommutatives Diagramm stellt graphisch dar, dass verschiedener Verkettungen von Abbildungen das gleiche Ergebnis liefern.

$$A \xrightarrow{f} B$$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "h"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$$h = g \circ f = g(f(x))$$

$$\begin{tikzcd} V \arrow[r, "\underline{\underline{A}}"] \arrow[rr, bend right, "\underline{\underline{C}}"'] & W \arrow[r, "\underline{\underline{B}}"] & U \end{tikzcd}$$

$$\underline{\underline{C}} = \underline{\underline{B}} \cdot \underline{\underline{A}}$$

Koordinatenabbildung:

$$\begin{tikzcd}[row sep=large, column sep=huge]
X \arrow[r, bend left, "F"] \arrow[d, shift right, "k_X"'] & Y \arrow[l, bend left, "F^{-1}"] \arrow[d, shift right, "k_Y"'] \\
\mathbb{R}^n \arrow[u, shift right, "k_X^{-1}"'] \arrow[r, bend left, "\underline{\underline{A}}"] & \mathbb{R}^m \arrow[u, shift right, "k_Y^{-1}"'] \arrow[l, bend left, "\underline{\underline{A}}^{-1}"]
\end{tikzcd}$$

Abbildungsmatrizen bei Koordinatenwahl/Basiswechsel:

$$\begin{tikzcd}[row sep=large, column sep=huge]
X \arrow[r, bend left, "F"] \arrow[d, shift right, "k_X"'] & Y \arrow[l, bend left, "F^{-1}"] \arrow[d, shift right, "k_Y"'] \\
\tilde{V} \arrow[u, shift right, "k_X^{-1}"'] \arrow[r, bend left, "\tilde{\underline{\underline{A}}}"] \arrow[d, shift right, "T"'] & \tilde{W} = \tilde{U} \arrow[u, shift right, "k_Y^{-1}"'] \arrow[l, bend left, "\tilde{\underline{\underline{A}}}^{-1}"] \arrow[d, shift right, "T'"'] \\
V \arrow[u, shift right, "T^{-1} = S"'] \arrow[r, bend left, "\underline{\underline{A}}"] & W = U \arrow[u, shift right, "S'"'] \arrow[l, bend left, "\underline{\underline{A}}^{-1}"]
\end{tikzcd}$$

Bsp: Beispiel 80 $\qquad F: \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \longmapsto \begin{bmatrix} 7x + 5y - 8z \\ 5x + 3y - 4z \\ -x - 3y + 8z \end{bmatrix}$$

a)

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 7 \\ 5 \\ -1 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 5 \\ 3 \\ -3 \end{bmatrix}$$

$$z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} -8 \\ -4 \\ 8 \end{bmatrix}$$

$$\Rightarrow \underline{\underline{A}} = \begin{bmatrix} 7 & 5 & -8 \\ 5 & 3 & -4 \\ -1 & -3 & 8 \end{bmatrix}$$

b)

$$\mathcal{B} = \{ b^{(1)} = e_1,\ b^{(2)} = e_1 + e_2,\ b^{(3)} = e_2 + e_3 \} \qquad \mathcal{B} \xrightarrow{T} \mathcal{E}$$

$$= \left\{ b^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ b^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ b^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$b^{(1)} = e_1 = x = 1 \cdot x + 0 \cdot y + 0 \cdot z$$

$$b^{(2)} = e_1 + e_2 = x + y = 1 \cdot x + 1 \cdot y + 0 \cdot z \qquad \Rightarrow T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$b^{(3)} = e_2 + e_3 = y + z = 0 \cdot x + 1 \cdot y + 1 \cdot z$$

$$b^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$b^{(2)} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$b^{(3)} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \mathcal{E} \cdot T = \mathcal{B}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

T^{-1}:

$$\left(\begin{array}{ccc|ccc} 1&1&0&1&0&0\\ 0&1&1&0&1&0\\ 0&0&1&0&0&1 \end{array}\right) \xrightarrow{\mathrm{II}-\mathrm{III}} \left(\begin{array}{ccc|ccc} 1&1&0&1&0&0\\ 0&1&0&0&1&-1\\ 0&0&1&0&0&1 \end{array}\right)$$

$$\xrightarrow{\mathrm{I}-\mathrm{II}} \left(\begin{array}{ccc|ccc} 1&0&0&1&-1&1\\ 0&1&0&0&1&-1\\ 0&0&1&0&0&1 \end{array}\right) = T^{-1}$$

c)

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "F"] \arrow[d, shift right, "k_x"'] & \mathbb{R}^3 \arrow[d, shift right, "k_x"'] \\ \mathcal{E} \arrow[u, shift right, "k_x^{-1}"'] \arrow[r, "\underline{\underline{A}}"] \arrow[d, shift left, "T^{-1}"] & \mathcal{E} = D \arrow[u, shift right, "k_x^{-1}"'] \arrow[d, shift left, "T^{-1}"] \\ B \arrow[u, shift left, "T"] \arrow[r, "\underline{\underline{B}}"] & B \arrow[u, shift left, "T"] \end{tikzcd}$$

$$\underline{\underline{B}} = \underline{\underline{T}}^{-1}\,\underline{\underline{A}}\,\underline{\underline{T}}$$

$$\underline{\underline{A}}\begin{bmatrix}1\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}7\\5\\-1\end{bmatrix} = \underbrace{a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}1\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\1\\1\end{bmatrix}}_{} =$$

$$\begin{bmatrix}1\\1\\0\end{bmatrix} \mapsto \begin{bmatrix}\vdots\end{bmatrix} =$$

$$\begin{bmatrix}0\\1\\1\end{bmatrix} \mapsto \begin{bmatrix}\vdots\end{bmatrix} =$$

<u>Verkettung:</u>

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, shift right, "k_V"'] & W \arrow[r, "g"] \arrow[d, "k_W"'] & U \arrow[d, "k_U"'] \\ \mathbb{R}^n \arrow[u, shift right, "k_V^{-1}"'] \arrow[r, "\underline{\underline{A}}"] \arrow[rr, bend right, "\underline{\underline{C}}"'] & \mathbb{R}^m \arrow[r, "\underline{\underline{B}}"] & \mathbb{R}^o \end{tikzcd}$$

$$C = B \cdot A$$

$$[\underline{v}]_{\mathbb{R}^o} = B \cdot \underbrace{A \cdot [v]_{\mathbb{R}^n}}_{[v]_{\mathbb{R}^m}}$$

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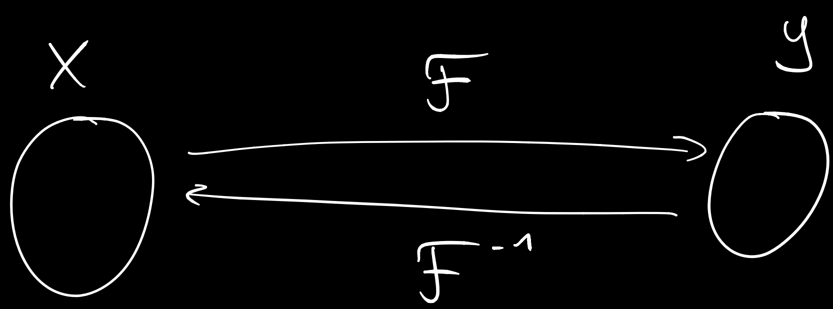
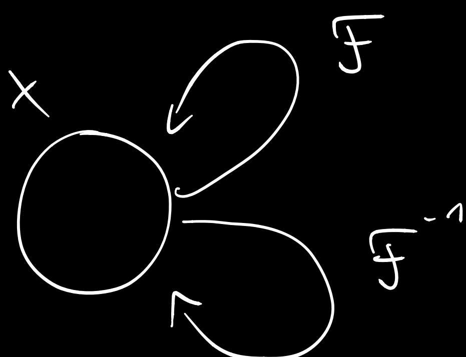
Isomorphismus / Automorphismus:



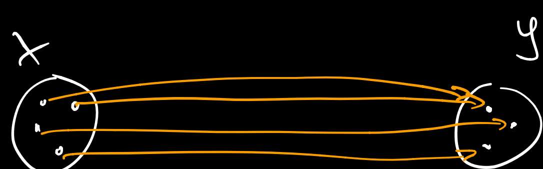


Repetition:

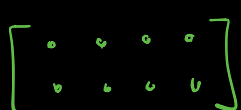
Injektivität:

$|Y| \geq |X|$

Surjektivität:



$|X| \geq |Y|$

Frage:  Isomorph? Automorph? Nichts

Isomorphe Abbildung $\Longleftrightarrow$ Quadr. Matrix mit vollem Rang

Abbildungsmatrizen / Darstellungsmatrizen lin. Abb.

Bsp.: Sei $V = \mathcal{P}_2$, $W = \mathcal{P}_1$, $F : V \to W$, $p(x) \mapsto p'(x) + p''(x)$

$$\begin{aligned}
1 &\xrightarrow{F} 0 = \boxed{0 \cdot 1 + 0 \cdot x} \\
x &\xrightarrow{F} 1 = 1 \cdot 1 + 0 \cdot x \\
x^2 &\xrightarrow{F} 2x + 2 = 2 \cdot 1 + 2 \cdot x
\end{aligned}
\quad \Rightarrow \quad \underline{A} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

neu

$$\begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ x \end{bmatrix}$$

Bsp:

$$\begin{tikzcd}
p(x) = x^2 + 4x - 7 \arrow[r, "F"] \arrow[d, shift right, "k_x"'] & 2x + 4 + 2 = \underline{2x+6} \arrow[d, shift right, "k_x"'] \\
\begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix} \arrow[r, "\underline{A}"] \arrow[u, shift right, "k_x^{-1}"'] & \begin{bmatrix} 6 \\ 2 \end{bmatrix} \arrow[u, shift right, "k_x^{-1}"']
\end{tikzcd}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} -7 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Kommutative Diagramme:

$$A \xrightarrow{f} B$$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "h"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Falls $h(x) = g \circ f(x) = g(f(x))$ $\Leftrightarrow$ das Diagramm kommutiert

Koordinatenabbildung:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
X \arrow[r, shift left, "f"] \arrow[d, shift right, "T \quad k_X"'] & Y \arrow[l, shift left, "f^{-1}"] \arrow[d, shift right, "k_Y"'] \\
V = \mathbb{R}^n \arrow[u, shift right, "k_X^{-1}"'] \arrow[r, shift left, "\underline{\underline{A}}"] & \mathbb{R}^m \arrow[u, shift right, "k_Y^{-1} \quad S"'] \arrow[l, shift left, "\underline{\underline{A}}^{-1}"]
\end{tikzcd}$$

Abbildungsmatrix unter Koordinatenwahl / Basiswechsel:

$$\begin{tikzcd}
 & X \arrow[rr, shift left, "F"] \arrow[d, shift right, "k_X"'] & & Y \arrow[ll, shift left, "F^{-1}"] \arrow[d, shift right, "k_Y"'] & \\
\mathbb{R}^n & \tilde{V} \arrow[rr, shift left, "\underline{\underline{\tilde{A}}}"] \arrow[u, shift right, "k_X^{-1}"'] \arrow[d, shift right, "\underline{\underline{T}}"'] & & \tilde{W} \arrow[ll, shift left, "\underline{\underline{\tilde{A}}}^{-1}"] \arrow[u, shift right, "k_Y^{-1}"'] \arrow[d, shift right, "\underline{\underline{T}}'"'] & \mathbb{R}^m \\
 & V \arrow[rr, shift left, "\underline{\underline{A}}"] \arrow[u, shift right, "\underline{\underline{T}}^{-1} = \underline{\underline{S}}"'] & & W \arrow[ll, shift left, "\underline{\underline{A}}^{-1}"] \arrow[u, shift right, "\underline{\underline{S}}'"'] &
\end{tikzcd}$$

Bsp: $F : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} 7x + 5y - 8z \\ 5x + 3y - 4z \\ -x - 3y + 8z \end{bmatrix}$$

a) $\mathcal{E} = \{\underline{e}_1, \underline{e}_2, \underline{e}_3\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xmapsto{F} \underline{\begin{bmatrix} 7 \\ 5 \\ -1 \end{bmatrix}} = \underline{7 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

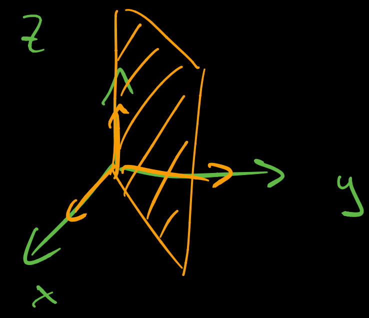
$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 5 \\ 3 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} -8 \\ -4 \\ 8 \end{bmatrix}$$

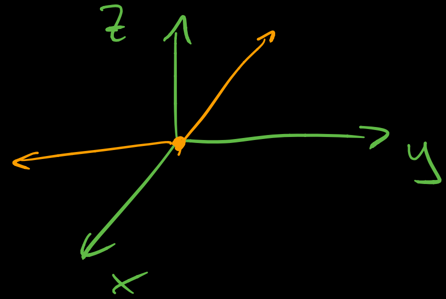
$$\Rightarrow \underline{\underline{A}} = \underline{\underline{\begin{bmatrix} 7 & 5 & -8 \\ 5 & 3 & -4 \\ -1 & -3 & 8 \end{bmatrix}}}$$

Nice to know:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



b) $\mathcal{B} = \{ b_1 = e_1, \ b_2 = e_1 + e_2, \ b_3 = e_2 + e_3 \}$

$$= \left\{ b_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ b_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ b_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right.$$

$$b_1 = \underline{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \underline{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \underline{1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$b_2 = \underline{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \underline{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} = \underline{1 \cdot \qquad 1 \cdot \qquad 0 \cdot}$$

$$b_3 = \underline{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} = \underline{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} = \underline{0 \cdot \qquad 1 \cdot \qquad 1 \cdot}$$

$$\Rightarrow \underline{T} = \underline{\underline{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}}$$

$$\begin{tikzcd} \mathcal{B} \arrow[d, shift right, "T"'] \\ \mathcal{E} \arrow[u, shift right, "T^{-1}"'] \end{tikzcd}$$

$$\underline{\underline{T^{-1}}}: \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & \emptyset & \\ 0 & 1 & 1 & \emptyset & 1 & \\ 0 & 0 & 1 & & & 1 \end{array}\right] \xrightarrow{\text{II}-\text{III}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{I}-\text{II}} \left[\begin{array}{ccc|ccc} 1 & & \emptyset & 1 & -1 & 1 \\ & 1 & & 0 & 1 & -1 \\ \emptyset & & 1 & 0 & 0 & 1 \end{array}\right] = \underline{\underline{T^{-1}}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

c)

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "F"] \arrow[d, shift right, "k_x"'] & \mathbb{R}^3 \arrow[d, shift right, "k_x"'] \\
\mathcal{E} \arrow[r, "\underline{\underline{A}}"] \arrow[u, shift right, "k_x^{-1}"'] \arrow[d, shift left, "\underline{\underline{T}}^{-1}"] & \mathcal{E} \arrow[u, shift right, "k_x^{-1}"'] \arrow[d, shift left, "\underline{\underline{T}}^{-1}"] \\
B \arrow[r, "\underline{\underline{C}}"'] \arrow[u, shift left, "\underline{\underline{T}}"] \arrow[r, bend left=60] & B \arrow[u, shift left, "\underline{\underline{T}}"]
\end{tikzcd}$$

$$\underline{\underline{C}} = \underline{\underline{T}}^{-1} \underline{\underline{A}}\, \underline{\underline{T}} =$$

<u>Verkettung lin. Abb.:</u>

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, shift right, "k_V"'] & W \arrow[r, "g"] \arrow[d, shift right, "k_W"'] & U \arrow[d, shift right, "k_U"'] \\
A \arrow[r, "\underline{\underline{M}}_A^B"] \arrow[u, shift right, "k_V^{-1}"'] \arrow[rr, bend right, "\underline{\underline{M}}_A^C"'] & B \arrow[r, "\underline{\underline{M}}_B^C"] \arrow[u, shift right, "k_W^{-1}"'] & C \arrow[u, shift right, "k_U"']
\end{tikzcd}$$

$\Rightarrow \; g \circ f = g(f(\cdot))$

$\Rightarrow \; \underline{\underline{M}}_A^C = M_B^C \cdot M_A^B$

$1, x, x^2, x^3, \ldots \;\rightarrow\; \mathcal{P}_n$

$B = \operatorname{span}\{1, t^2, t^4\}$ $\qquad 1, t^2, t^4 \;\leftrightarrow\; 1, t, t^2, t^3, t^4$

$\tilde{B} = \operatorname{span}\{t, t^3\}$